Solution 9

1. Consider the problem of minimizing $f(x,y,z) = (x+1)^2 + y^2 + z^2$ subjecting to the constraint $g(x,y,z) = z^2 - x^2 - y^2 - 1,\ z > 0$. First solve it by eliminating z and then by Lagrange multipliers.

   Solution. Old method. From $g = 0$ get $z^2 = x^2 + y^2 + 1$. Plug in f to get $h(x,y) = (x+1)^2 + y^2 + x^2 + y^2 + 1$. When (x_0, y_0, z_0) is a local minimizer of f subject to $g = 0$, (x_0, y_0) is a local minimizer of $h(x,y)$. Hence $h_x = h_y = 0$ at (x_0, y_0) which yields

   $$2(x+1) + 2x = 0, \quad 2y + 2y = 0 \ ,$$

   so $x = -1/2, y = 0$. We conclude that $(-1/2, 0, \sqrt{5}/2)$ is a critical point and hence a candidate for the local minimizer. (With further reasoning, it is really a global minimizer.)

   New method, there is some λ such that

   $$x + 1 = \lambda x,\ y = \lambda y,\ z = -\lambda z,\ z^2 - x^2 - y^2 = 1 \ .$$

   The fourth equation implies that z is positive, so the third equation yields $\lambda = -1$. Then we get $x = -1/2, y = 0$ and $z = \sqrt{5}/2$.

   Note. Usually we don't have to check the condition $\nabla g \neq (0,0,0)$ before applying the theorem on Lagrange multipliers. You may check it if you like when everything is done.

2. Let $f, g_1, \cdots, g_m$ be C^1-functions defined in some open U in $\mathbb{R}^{n+m}$. Suppose (x_0, y_0) is a local extremum of f in $\{(x,y) \in U : \ g_1(x,y) = \cdots = g_m(x,y) = 0\ \}$. Assuming that $D_y G(x_0, y_0)$ is invertible where $G = (g_1, \cdots, g_m)$, show that there are $\lambda_1, \cdots, \lambda_m$ such that

   $$\nabla f + \lambda_1 \nabla g + \cdots + \lambda_m \nabla g_m = 0 \ ,$$

   at (x_0, y_0).

   Solution. Similar to the special case $f(x,y,z)$ over $g(x,y,z) = 0$. What we need is a statement from linear algebra: Let E be an n-dimensional subspace of R^{n+m} and $u_1, \cdots, u_m$ are m-many independent vectors perpendicular to E. Then for any w perpendicular to E, $w + \sum_{j=1}^m \lambda_j u_j = 0$. Proof: Pick an orthonormal basis of E, $v_1, \cdots, v_n$. Then $v_1, \cdots, v_n, u_1, \cdots, u_m$ form a basis of $\mathbb{R}^{n+m}$. So

   $$w + \mu_1 v_1 + \cdots + \mu_n v_n + \lambda_1 u_1 + \cdots + \lambda_m u_m = 0 \ .$$

   Taking inner product with v_k, we get $0 = w \cdot v_k + \mu_k = \mu_k$ for all k. Hence $w + \sum_{j=1}^m \lambda_j u_j = 0$.

3. Solve the IVP for $f(t,x) = \alpha t(1+x^2), \alpha > 0,\ t_0 = 0$, and discuss how the (largest) interval of existence changes as α and x_0 vary.

   Solution. The solution is given by

   $$x(t) = \tan(\tan^{-1} x_0 + \alpha t^2/2) \ ,$$

   where the tangent function is chosen so that $\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$. The (maximal) interval of existence is $(-a, a)$ where

   $$\sqrt{a} = \frac{1}{\alpha}(\pi - 2\tan^{-1} x_0) \ .$$

   We see that for fixed α, the interval shrinks as x_0 increases, and for fixed x_0, it shrinks too as α increases. The maximal interval of existence depends on f, t_0 and x_0 in a complicated manner.

4. Let $f \in C(R)$ where R is a closed rectangle. Suppose x solves $x' = f(t,x)$ for $t \in (a,b)$ with $(t, x(t)) \in R$. Show that x can be extended to be a solution in $[a,b]$.

   Solution. First, since $(t, x(t))$ remains in R which is bounded, there is $\{t_n\}, t_n \to b^-$ such that $x(t_n) \to z$ for some z. We claim in fact $x(t) \to z$ as $t \to b^-$. For $\varepsilon > 0$, take δ to satisfy $\delta < \varepsilon/(2M), M = \sup_R |f|$. Then for $t, b - t < \delta$, we can find some $t_n \in (t,b)$ such that $|x(t_n) - z| < \varepsilon/2$. Then

$$\begin{aligned} |x(t) - z| &\leq |x(t) - x(t_n)| + |x(t_n) - z| \\ &< \left| \int_{t_n}^t f(s, x(s))\, ds \right| + \frac{\varepsilon}{2} \\ &\leq M|t_n - t| + \frac{\varepsilon}{2} \\ &\leq \varepsilon\ . \end{aligned}$$

   By defining $x(b) = z$, we see that $x(t)$ is continuous on $(a,b]$. In the relation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))\, ds, \quad t \in (a,b)\ ,$$

   we can let $t \to b^-$ to show it remains true in $(a,b]$. Similarly, we can show the solution extends to $[a,b]$ too.

5. Let $f \in C(R)$ where R is a closed rectangle satisfy a Lipschitz condition in R. Suppose that x solves $x' = f(t,x)$ for $t \in [a,b]$ where $(b, x(b))$ lies in the interior of R. Show that there is some $\delta > 0$ such that x can be extended as a solution in $[a, b+\delta]$.

   Solution. Solve the IVP of the equation passing the point $(b, x(b))$. Since this point lies in the interior of R, we can find a small rectangle R_1 inside R taking this point as the center. By applying the Picard-Lindelof theorem to R_1 we obtain a solution extending beyond b. By uniqueness it coincides with the old solution in their common interval of existence.

6. Provide a proof to Theorem 3.15 (Picard-Lindelof theorem for systems).

   Solution The proof is basically the same as in the equation case. Tutor will do it in class.

7. Let $\mathbf{e}_n = (0, \cdots, 0, 1, 0, \cdots,)$ by the sequence with 1 at the n-th place and equal to 0. Consider the sequence formed by these $\mathbf{e}_j$'s. Show that it has no convergent subsequences in the space l^p, $1 \leq p \leq \infty$ Recall that l^p is the space consisting of all sequences $\mathbf{a} = \{a_n\}$ satisfying $\|\mathbf{a}\|_p = (\sum_n |a_n|^p)^{1/p} < \infty$ and $\|\mathbf{a}\|_\infty = \sup_n |a_n|$.

   Solution. Suppose on the contrary this sequence has a limit $\mathbf{a} = \{a_n\}$. (I have used bold letters to denote sequences.) Then $\lim_{n\to\infty} \|\mathbf{e}_n - \mathbf{a}\| = 0$. From the definition of the l^p-norm it means every component of $\mathbf{e}_n - \mathbf{a}$ tends to zero. Since the k-component of $\mathbf{e}_n$ becomes zero when $n > k$, the sequence $\mathbf{a}$ must be the zero sequence. Therefore, in case the sequence formed by $\mathbf{e}_n$'s has a convergent subsequence, it also converges to the zero sequence in the l^p-norm. But this is impossible since $\lim_{n\to\infty} \|\mathbf{e}_{n_k} - \mathbf{0}\|_p = 1$.

8. Consider $\{f_n\}, f_n(x) = x^{1/n}$, as a subset $\mathcal{F}$ in $C[0,1]$. Show that it is closed, bounded, but has no convergent subsequence in $C[0,1]$.

   Solution. It means $\mathcal{F}$ is not precompact. $\mathcal{F}$ is bounded as $\|f_n\|_\infty \leq 1$ for all $f \in \mathcal{F}$. Next, we claim that it has no convergent subsequence. Suppose on the contrary there is one subsequence $\{f_{n_j}\}$ converges to some $g \in C[0,1]$. Then, for each x, one must have

$\lim_{j\to\infty} f_{n_j}(x) = g(x)$. However, it is clear that the pointwise limit of f_n is the function $f(x) = 1, x \in (0,1]$ and equals 0 at $x = 0$. So g must coincide with f, but this is impossible as g is continuous on $[0,1]$ but f is discontinuous at $x = 0$.

We still need to check that $\mathcal{F}$ is closed. Let $\{h_n\}$ be a sequence in $\mathcal{F}$ converging to some $h \in C[0,1]$. Consider two cases. First, this sequence contains infinitely many distinct functions. Then we can extract a subsequence from it which is also a subsequence of $\{f_n\}$. As above we see that this is impossible because h is continuous but f is not. Second, $\{h_n\}$ contains only finitely many functions. Then one function, say, f_{n_0}, appears infinitely many times. We can take a subsequence $\{h_{n_j}\}$ consisting of the single f_{n_0}. It must be true that $h = f_{n_0} \in \mathcal{F}$. We conclude that $\mathcal{F}$ is a closed set.

9. Prove that $\{\cos nx\}_{n=1}^{\infty}$ does not have any convergent subsequence in $C[0,1]$.

   Solution. By Arzela Theorem it suffices to show that this sequence has no subsequence that is equicontinuous. Suppose on the contrary, given $\varepsilon > 0$, there exists some $\delta > 0$ such that
   $$|\cos n_k x - \cos n_k y| < \varepsilon, \quad \forall k \geq 1,\ x, y,\ |x - y| < \delta.$$
   Now, take $\varepsilon = 1$ so δ is fixed. Take $x = 0$ and $y = \pi/n$. When n is large $|0 - \pi/n| < \delta$, one should have $|\cos n0 - \cos n\pi/n| < \varepsilon = 1$. But actually we have $|\cos n0 - \cos n\pi/n| = 2$, contradiction holds.

10. Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous.

    Solution. Recall that any continuous function in $\overline{G}$ is uniformly continuous. (The proof is similar to the special case $C[a,b]$.) Now, let the finite set be $\{f_1, \cdots, f_N\}$. Since each f_k is uniformly continuous, for $\varepsilon > 0$, there is some δ_k such that $|f_k(x) - f_k(y)| < \varepsilon$ for all $x, y,\ |x - y| < \delta_k$. If we take $\delta = \min\{\delta_1, \cdots, \delta_N\}$. Then $|f_k(x) - f_k(y)| < \varepsilon$ for $x, y,\ |x - y| < \delta$ and all k. On the other hand, it is clearly bounded by the maximum of $\|f_1\|_\infty, \cdots, \|f_N\|_\infty$.